A STATISTICAL METHOD IN THE THEORY OF STABILITY OF SHELLS

(STATISTICHESKII METOD V TEORII USTOICHIVOSTI OBOLOCHEK)

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1. We consider a shell, subjected to a loading which increases proportionally to some parameter λ. We assume that the boundary conditions of the shell admit for $\lambda = 1$ a membrane state of stress. In this case, for a large class of shells, the following picture of the change of the number of equilibrium shapes of the shell and their properties will be typical.

There exists a certain number λ_0 such that for $\lambda < \lambda_0$ there exists a unique membrane equilibrium shape of the shell which corresponds to an absolute minimum of the energy of the system, composed of the shell and external forces. Further, there exists a number $\lambda_1 \geqslant \lambda_0$ such that for $\lambda_0 \leqslant \lambda < \lambda_1$ the shell possesses in addition to the membrane equilibrium shape also a bending equilibrium shape, however, the membrane equilibrium shape of the shell will have a lower energy level than an arbitrary bending state. Further, there exists a number $\lambda_{00} \geqslant \lambda_1$ such that for $\lambda_1 < \lambda < \lambda_{00}$ the membrane equilibrium shape of the shell, even though it is associated with a relative minimum of the energy, is accompanied by at least one bending equilibrium shape, associated with lower energy level. Finally, for $\lambda > \lambda_{00}$ the membrane equilibrium shape of the shell, in general, ceases to be associated with an energy minimum.

Such a change of equilibrium shapes was factually established in a series of investigations (see the bibliographies in [1,2]) for a spherical and cylindrical shell on the basis of application of approximate methods. The same finding received a strong justification for a rather large class of shells and boundary conditions [3].

From what has been mentioned above, it becomes clear that even if it were possible to completely surmount all mathematical difficulties associated with the solution of the basic equations of the nonlinear theory of shells, the problem could still not be considered as being solved completely, since the degree of reality of each of the possible equilibrium shapes of the shell for $\lambda_0 < \lambda < \lambda_{00}$ would still not be determined.

The selection of the most real equilibrium shape of the shell must be made, taking additional considerations into account. It is rational to introduce as a measure of reality of a certain equilibrium shape of the shell the probability of finding the shell in that shape.

The idea of introducing the probability considerations into stability problems of shells was expressed by Feodos'ev [4] and Vol'mir [1]. The introduction of probability considerations in our opinion will significantly advance the solution of such important questions as:

(1) Determination of admissible loads on the shell in studying stability, taking into account the conditions of its behavior and the irregularities in its manufacture;

(2) The determination of inaccuracies with regard to basic shell parameters. Of most importance in this regard the most important factor we have in mind is the analysis of the necessary accuracy in manufacturing the middle surface of the shell.

The development of the statistical theory of stability of shells must in our opinion include the following items:

(1) Methods of statistical description of factors which determine the random characteristics of deformation of the shell. Methods and technique of experimental determination of statistical characteristics of indicated factors.

(2) Methods of statistical description of parameters which characterize the deformation of the shell. Methods and techniques of experimental determination of statistical characteristics of indicated parameters.

(3) Relations between the statistical characteristics of the parameters, which describe the deformation of the shell and the statistical characteristics of factors which determine the accidental character of the deformation of the shell.

An approximate approach to the construction of such a theory is presented below.

Let us assume that all factors which determine the accidental character of shell bending may be subdivided into three groups:

(1) The dispersion of elastic and geometric shell properties;

(2) The dispersion of parameters characterizing the method of shell fixing.

(3) The dispersion of external loading applied to the shell.

Further, even though the indicated groups may contain also functional parameters, as for example, the deviation on the shape of the middle surface of the shell, deviations in the thickness of the shell, etc., we shall still assume that the totality of factors of the first two groups may be described by a finite number of parameters $a_1, \ldots, a_m$. In view of this it is natural to assume that the probability properties of the first two groups of factors will be given, if the law $\phi(a_1, \ldots, a_m)$ of distribution of the parameters is specified. We now assume that the parameters $a_1, \ldots, a_m$ are fixed and we now write down the equations of shell motion, subjected to a loading $F(P, t)$ and taking energy dissipation into account. We have

$$\rho w_{tt} + 2\gamma w_t + D\nabla^4 w = \Phi_{yy}(w_{xx} + f_{xx}) + \Phi_{xx}(w_{yy} + f_{yy}) -$$

$$- 2\Phi_{xy}(w_{xy} + f_{xy}) + Z(P, t) \qquad (1.1)$$

$$\nabla^4\Phi = 2Eh(w_{xy}^2 - w_{xx}w_{yy} - f_{xx}w_{yy} - f_{yy}w_{xx} + 2f_{xy}w_{xy}) \qquad (1.2)$$

In these equations ρ is the mass density of the shell, referred to a unit area of the middle surface of the shell; the energy dissipation in the shell is described by the term $2\gamma w_t$. For the sake of simplicity in the equations (1.1) and (1.2), we neglected the energy of longitudinal motions of the shells and we assume that $F(P, t)$ has only one component $Z(P, t)$. All these assumptions may be omitted at the cost of certain complication of further calculations.

We assume that w satisfies certain homogeneous support conditions and further

$$\Phi|_\Gamma = r(s), \qquad \frac{\partial\Phi}{\partial n} = q(s) \qquad (1.3)$$

where $r(s)$, $q(s)$ are certain functions of the arc contour length s.

We shall seek an approximate solution of the problem in the form:

$$w = \sum_{k=1}^{n} q_k(t)\chi_k(P) \qquad (1.4)$$

Here $\chi_k(P)$ is the base in the energy space of shell bending [5,6]. To determine $q_k(t)$ we use the method of Bubnov-Galerkin, assuming that χ_k are also normal in L_2. We then obtain the following system:

$$\ddot{q}_k + \frac{2\gamma}{\rho}\dot{q}_k = -\frac{1}{\rho}\frac{\partial U}{\partial q_k} + \frac{1}{\rho}Z_k(t) \quad (k = 1, \ldots, n), \quad \left(Z_k = \int_\Omega Z(P, t)\chi_k(P)\,dP\right) \qquad (1.5)$$

Here U is the potential energy of shell deformation expressed in terms of q_k.

The system (1.5) may be considered as the equations of motion of a certain point moving in n-dimensional space of the coefficients $q_1, \ldots, q_n$. This point moves in the force field with the potential $\rho^{-1} U$ and under the action of accidental loadings $\rho^{-1} Z_k(t)$. We shall assume below that

$$Z(P,t) = Z^{(1)}(P,t) + Z^{(2)}(P,t) + Z^{(3)}(P,t) \qquad (Z^{(1)}(P,t) = \text{м. о.}\, Z(P,t)) \tag{1.6}$$

Here $Z^{(2)}(P, t)$ is the fluctuation term producing accelerations of the point of the type of Brownian motion, $Z^{(3)}(P, t)$ representing the continuous random process.

We assume further that we can set with sufficient accuracy

$$Z^{(3)}(P,t) = \sum_{k=1}^{n} \sum_{l=1}^{n_1} a_{kl}\, \chi_k(P)\, \psi_l(t) \tag{1.7}$$

Here $\psi_l(t)$ are some fixed functions of time. We shall assume that a continuous random process is given if the law of distribution $\theta(a_{kl})$ of parameters a_{kl} is known. In accordance with (1.6) we have

$$Z_k(t) = Z_k^{(1)}(t) + Z_k^{(2)} + \sum_{l=1}^{n_1} a_{kl}\, \psi_l(t) \tag{1.8}$$

The problem is now reduced to finding the law of distribution of $q_1, \ldots, q_n$ in time.

To solve this problem we assume that the groups of parameters $a_1, \ldots, a_m$ and a_{kl} and the random process $Z^{(2)}(P, t)$ are statistically independent. We assume further that the parameters $a_1, \ldots, a_m$, a_{kl} have acquired a certain fixed value and we seek the law of distribution of $q_1, \ldots, q_n$ under these restrictions. For the instants of time $t > \rho/\gamma$ the distribution law to be found may be determined from the Smolukhovskii equation [7]

$$\frac{\partial f}{\partial t} = \sum_{i=1}^{n} \frac{\partial}{\partial q_i} \left\{ \left[\frac{\partial U}{\partial q_i} - Z_i^{(1)}(t) - \sum_{l=1}^{n_1} a_{kl}\, \psi_l \right] f \right\} \frac{1}{2\gamma} + \frac{\delta \rho^2}{4\gamma^2} \sum_{i=1}^{n} \frac{\partial^2 f}{\partial q_i^2} \tag{1.9}$$

In equation (1.9) the parameter δ characterizes the dispersion of collisions (impacts), acting upon the shell; the smaller δ, the smaller the dispersion of impacts applied to the shell. The parameter δ characterizes the conditions under which the shell is put into service, and must be determined from experience.

In as much as f is a certain distribution law, equation (1.9) must be supplemented by the following conditions valid for $t > 0$.

$$1)\ f \geqslant 0, \qquad 2)\ \int_{-\infty}^{+\infty} \cdots \int f\, dq_1, \ldots, dq_n = 1, \qquad 3)\ f \to 0 \ \text{при}\ q_1^2 + \ldots + q_n^2 \to \infty \tag{1.10}$$

Furthermore, $f(q_1, \ldots, q_n, 0) = f^*(q_1, \ldots, q_n)$, where f^* is the law of distribution of $q_1, \ldots, q_n$ initially.

Let us suppose that we were successful in finding f from (1.9), (1.10), Obviously f will depend also on the parameters $a_1, \ldots, a_m, a_{kl}$; thereby, the unconditional distribution law f° for the conditions considered will be

$$f^\circ(q_1, \ldots, q_n, t) = \int_{-\infty}^{+\infty} \cdots \int f(q_1, \ldots, q_n, t, a_k, a_{kl})\, \varphi(a_k)\, \theta(a_{kl})\, da_k\, da_{kl} \qquad (1.11)$$

Let us investigate several important cases in which the realization of the scheme indicated above is possible to the end and in which explicit formulas may be obtained.

Let $Z^{(3)} \equiv 0$, and let $Z^{(1)}$ be independent of time. In this case the distribution $f(q_1, \ldots, q_n)$ which will be approached as $t \to \infty$, must be determined from the equation

$$\frac{\delta\rho^2}{2\gamma} \sum_{i=1}^{n} \frac{\partial^2 f}{\partial q_i^2} + \sum_{i=1}^{n} \frac{\partial}{\partial q_i} \left[\left(\frac{\partial U}{\partial q_i} - Z_i^{(1)} \right) f \right] = 0 \qquad (1.12)$$

It can be easily verified that the function

$$\frac{1}{J} \exp\left[\left(-U + \sum_{k=1}^{n} q_k Z_k^{(1)} \right) \frac{2\gamma}{\delta\rho^2} \right] \qquad (1.13)$$

$$J = \int_{-\infty}^{+\infty} \cdots \int \exp\left[\left(-U + \sum_{k=1}^{n} q_k Z_k^{(1)} \right) \frac{2\gamma}{\delta\rho^2} \right] dq_1 \ldots dq_n$$

satisfies all the conditions (1.10) and the equation (1.9). The distribution (1.13) is seen to be the Gibbs distribution.

The conditionless law of distribution in accordance with (1.13) is determined by formula

$$f^\circ(q_1, \ldots, q_n) = \int_{-\infty}^{+\infty} \cdots \int f(q_1, \ldots, q_n, a_1, \ldots, a_n, a_{kl})\, \varphi(a_k)\, \theta(a_{kl})\, da_k da_{kl} \qquad (1.14)$$

The magnitude f° may be taken as a measure of reality of a specific equilibrium shape of the shell.

Formula (1.14) yields a sufficiently complete solution.

Let us note certain most significant features of the method of statistical analysis of shell equilibrium advanced in this paper.

1. The calculation of the distribution law in accordance with the formula (1.14) requires neither a preceding solution of the problem of shell equilibrium under specific loading, nor the analysis of the

number of equilibrium shapes, nor the replacement of real relations between the deflections and the external load by single-valued functions, etc. It is only required to note the expression of the potential energy of the system in terms of generalized coordinates. The construction of this expression however, does not present any difficulty.

2. The analysis of the distribution law by formula (1.14) is reduced to quadratures. Thereby, since the expressions under the integral in formula (1.14) are sufficiently smooth functions, these quadratures may by evaluated numerically without any complications, even in cases in which, for the sake of accuracy, a large number of parameters $q_1, \ldots, q_n$ are used. In this connection, obviously, no special difficulties arise, associated with the use of machines for the calculations by formula (1.14).

3. Formula (1.11) takes into account in principle all basic factors which determine the accidental character of shell bending, among them also such accidental loads as change rather rapidly with time, and also loads which change periodically with a period comparable to the period of variation of the shell itself, etc. This formula makes it possible to follow up the process of probability changes in time. It is true that to this end the solution of the corresponding boundary value problem for equation (1.9) is required first.

However, equation (1.9) is one of those equations which are particularly suitable for numerical evaluation.

2. Let us consider the stability of a quadratic cylindrical panel subjected to the action of a longitudinal compressive force (Fig. 1). In solving this problem, we assume that accidental deviations in the shape of the middle surface and the action of accidental, rapidly varying external loads are taken into account.

The potential energy of the shell may be taken in the form [1]

$$U = \frac{\pi^4 E h^4}{8a^2}\left\{\left(\zeta^4 + 4\zeta^3\zeta_0 + 4\zeta_0\zeta_{\mathrm{H}}^2\right) - \frac{64k}{\pi^4}\left(\frac{5\zeta^2}{9} + \zeta^2\zeta_0\right) + \frac{16\zeta^2}{\pi^2}(S_{\mathrm{B}} - S) - \frac{32}{\pi^2}S\zeta\zeta_0\right)\right\} \tag{2.1}$$

$$\zeta = \frac{f}{2h}, \quad \zeta_0 = \frac{f_0}{2h}, \quad k = \frac{a^2}{2Rh}, \quad S_{\mathrm{B}} = 3.6 + \frac{k^2}{39.5}, \quad S = \frac{Qa^2}{4Eh^2}$$

Here a is the edge length of the square of the shell, $2h$ is the shell thickness, E Young's modulus, f is the shell deflection, f_0 is the initial deflection of the shell.

We shall consider a shell with a curvature parameter $k = 12$. In this case the potential energy will be expressed by the formula

$$U = \frac{\pi^4 E h^4}{8a^2}\{\zeta^4 + \zeta^3(4\zeta_0 - 4.36) + \zeta^2[-3.86\zeta_0 + 11.85(1 - P)] - 23.41\,\zeta\zeta_0 P\}$$
$$P = S/S_B \tag{2.2}$$

In accordance with (1.13), the conditional distribution law ζ (the distribution law for a determined ζ_0) is given by the relation

$$f(\zeta, \zeta_0) = \frac{1}{J} e^{-\mu V(\zeta)}, \quad J = \int_{-\infty}^{\infty} e^{-\mu V(\zeta)}\, d\zeta, \qquad \mu = \frac{\pi^4 E h^4 \gamma}{4a^2\rho^2\delta} \tag{2.3}$$

In equations (2.3) the following notation is introduced

$$V(\zeta) = \zeta^4 + \zeta^3(4\zeta_0 - 4.36) + \zeta^2[-3.86\,\zeta_0 + 11.85(1 - P)] - 23.41\zeta\zeta_0 P$$

The conditionless distribution law will be given by the formula

$$f^\circ(\zeta) = \int_{-\infty}^{\infty} f(\zeta, \zeta_0)\,\varphi(\zeta_0)\, d\zeta_0 \tag{2.4}$$

where $\phi(\zeta_0)$ is the law of distribution of ζ_0.

Let us determine, for example, with the aid of (2.4), the probability of displacement ζ not exceeding unity in magnitude.

Obviously,

$$p = \int_{-1}^{+1} f^\circ(\zeta)\, d\zeta = \int_{-1}^{+1}\int_{-\infty}^{\infty} f(\zeta, \zeta_0)\,\varphi(\zeta_0)\, d\zeta_0\, d\zeta \tag{2.5}$$

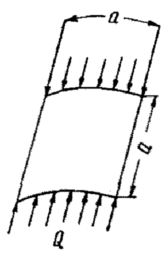

Fig. 1.

The results of numerical calculations by the above formulas are indicated in Figs. 1, 2, and 3, for the case when ζ_0 is subjected to a triangular symmetric distribution law.

Figure 4 indicates the dependence $p(D\,\zeta_0)$, where $D\,\zeta_0$ is the dispersion of ζ. The calculations were made for the $P = 0.5$ (that is, for the case in which the compressive force has half the upper critical value) and for $\mu = 1, 0.5, 0.2, 0.1$. The parameter μ for a fixed shell depends on δ, the quantity which characterizes the condition of shell performance. The larger the δ, the "quieter" are the conditions of shell performance. As seen from Fig. 4, for sufficiently small μ, that is, for not very "quiet" conditions of shell performance, $D\,\zeta_0$ has practically no influence on p.

Figure 3 indicates the dependence $p(D\,\zeta_0)$ for $\mu = 1$ and for different P. It can be seen from Fig. 3 that $p(D\,\zeta_0)$ has a different character for different P. If $P < 0.544$ of the lower critical number for the given case, then an increase in $D\zeta_0$ leads to a decrease of p. However, if $P > 0.544$,

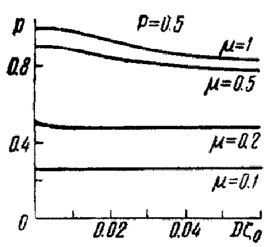

Fig. 2.

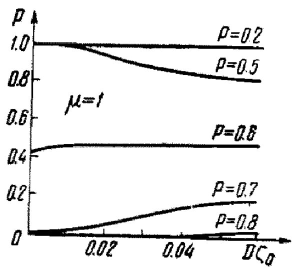

Fig. 3.

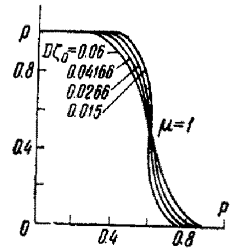

Fig. 4.

then an increase of $D\zeta_0$ is followed by an increase in p. This circumstance, which seems to be paradoxical at the first glance, is fully explicable.

In fact, a detailed analysis of the number of the equilibrium shapes of the shell and the degree of their stability indicates, that for $P > P_0$ (P_0 is the lower critical loading) and for large positive ζ_0 there exists a unique equilibrium shape which corresponds to ζ, being located outside the interval $[-1, +1]$.

For small positive ζ_0 the shell has three equilibrium shapes, and one of these is located within the portion $[-1, +1]$.

However, this shape corresponds to a higher level of potential energy of the shell as compared to the shapes which are located outside $[-1, +1]$. Therefore, even though for small positive ζ_0 there exist equilibrium shapes within $[-1, +1]$, they contribute but little to the increase in the probability of realizing the inequality $|\zeta| < 1$. For negative ζ_0 however, positive equilibrium positions also exist, which correspond to ζ in the segment $[-1, +1]$. But for negative ζ_0 these are precisely the forms which appear to be most stable and the larger the ζ_0, the more stable is the corresponding shape. Therefore, as we decrease the dispersion ζ_0, decreasing thereby the probability of occurrence of sufficiently large negative ζ_0, then the probability p may decrease.

If $P < 0.544$, then to each ζ_0 there corresponds a unique equilibrium shape of the shell and the smaller the ζ_0, the smaller the value of ζ, corresponding to the equilibrium shape of the shell. Thereby, obviously, by decreasing the dispersion ζ_0 the magnitude p must increase.

We also note that if $D\zeta_0$ is decreased, concentrating the distribution law ζ_0 on the negative ζ_0, then we will always have an increase of p. Therefore, it is natural to raise the question of introducing of technological, constructional and other types of measures, with the aid of which an artificial dispersion could be created, concentrating the distribution law of ζ_0 on negative values.

Figure 4 indicates the dependence of p on P for different $D\,\zeta_0$. It may be noted that the function $p(P)$ undergoes a sharp change for those values of the loading which are slightly higher than the lower critical number. These values of the loading are characterized by the circumstance that three equilibrium shapes of the shell correspond to them, and two stable equilibrium shapes of the shell have equal levels of potential energy.

The graphs in Fig. 4 are constructed for $\mu = 1$ and consequently they may be used only for the condition of shell performance corresponding to $\mu = 1$. However, it is entirely possible to construct a series of such graphs for different μ. This would give the possibility, using a given probability level of finding the shell in a specific state for given performance conditions, of determining the allowable dispersion in the shape of the middle surface of the shell. We turn our attention now to the analysis of the probability of snap-through of the shell. Figure 5 indicates a graph of the potential energy of the system composed of the shell and the external forces. For a certain value of $P > 0.544$ the snap-through of the shell will occur, if, as a consequence of accidental impacts, the potential barrier ζ_* is overcome. Therefore, it may be assumed that for a fixed ζ_0 the probability of snap-through p_* will be given by the relation

$$p_* = \int_{\zeta_*}^{\infty} f(\zeta, \zeta_0)\, d\zeta \qquad (2.6)$$

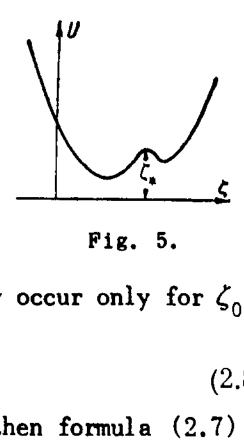

Fig. 5.

Applying the theorem on total probability, we obtain the following formula for the calculation of snap-through probability:

$$p_{1*} = \int_{\infty}^{\infty} p_*(\zeta_0)\, \varphi(\zeta_0)\, d\zeta_0 \qquad (2.7)$$

Further, if it is considered that snap-through may occur only for ζ_0, which satisfies the inequality

$$\zeta_0 \leqslant \zeta_{0*}(P) \qquad (2.8)$$

where ζ_0 is a certain number determined for each P, then formula (2.7) may be written in the form

$$p_{1*} = \int_{-\infty}^{\zeta_{0*}(P)} \int_{\zeta_*}^{\infty} f(\zeta, \zeta_0)\, \varphi(\zeta_0)\, d\zeta\, d\zeta_0 \qquad (2.9)$$

The results of calculations by formula (2.9) are given in Fig. 6. The circumstance should be pointed out here that as, $D\,\zeta_0$ increases, the snap-through probability decreases. This is explained by the fact that by decreasing $D\,\zeta_0$ we make large values of ζ_0 (in magnitude) less probable (we recall that the distribution law was assumed to be symmetrical). But for large positive ζ_0 snap-through does not occur, and for large negative ζ_0 snap-through is improbable, because the state of equilibrium before snap-through is associated in this case with a lower energy level than the state after snap-through.

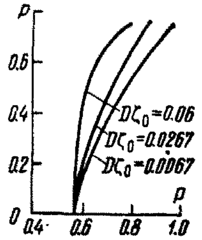

Fig. 6.

We note in conclusion that in employing the scheme outlined above, it is abvious that all possible practical operating conditions of shells must be divided in accordance with the level "quietness" of performance, and for each case to be analysed, the value of μ must be determined experimentally.

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